

Coexistence and Extinction in Time-Periodic Volterra-Lotka Type Systems with Nonlocal Dispersal

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Abstract. This paper deals with coexistence and extinction of time periodic Volterra-Lotka type competing systems with nonlocal dispersal. Such issues have already been studied for time independent systems with nonlocal dispersal and time periodic systems with random dispersal, but have not been studied yet for time periodic systems with nonlocal dispersal. In this paper, the relations between the coefficients representing Malthusian growths, self regulations and competitions of the two species have been obtained which ensure coexistence and extinction for the time periodic Volterra-Lotka type system with nonlocal dispersal. The underlying environment of the Volterra-Lotka type system under consideration has either hostile surroundings, or non-flux boundary, or is spatially periodic.

Key words. Time periodic Volterra-Lotka system, nonlocal dispersal, coexistence, extinction.

Mathematics subject classification. 45C05, 45G10, 45M20, 47G10, 92D25.

1 Introduction

Several models have already been derived or have still being derived to connect mathematics with ecology by many mathematicians and ecologists. Among them Volterra (1860-1940) and Lotka (1880-1949) are the two who contributed a model (in 1925) which is well known as competition model of two species, i.e.

$$\begin{cases} u_t = u(a_1 - b_1 u - c_1 v), \\ v_t = v(a_2 - b_2 u - c_2 v). \end{cases} \quad (1.1)$$

Since then it has drawn special attention of many mathematicians and ecologists on the following diffusive Volterra-Lotka type two species competition system,

$$\begin{cases} u_t = \nu_1 \Delta u + u(a_1(t, x) - b_1(t, x)u - c_1(t, x)v), & x \in D \\ v_t = \nu_2 \Delta v + v(a_2(t, x) - b_2(t, x)u - c_2(t, x)v), & x \in D \\ Bu = Bv = 0, & x \in \partial D, \end{cases} \quad (1.2)$$

where ν_1, ν_2 are positive constants, a_i, b_i, c_i ($i = 1, 2$) are positive smooth functions, $D \subset \mathbb{R}^N$ is a smooth bounded domain, and $Bu = Bv = 0$ are proper boundary conditions. Ecologically, the functions a_1, a_2 represent the respective growth rates of the two species, b_1, c_2 account for self-regulation of the respective species, and c_1, b_2 account for competition between the two species. Among those literatures on (1.2), many were published during 1980s (see [9], [15], [27], [28], etc.) and 2000s (see [5], [18], [20], [22], etc.).

The differential operator $u \mapsto \Delta u$ in (1.2) describes the random movements of individuals between adjacent locations and is therefore also referred to as a *random dispersal operator*. In reality, interactions or movements of individuals of the underlying systems occur between adjacent as well as non-adjacent locations. Certain integral operators, which are referred to as *nonlocal dispersal operators*, are used to describe nonlocal interactions of individuals in ecology (see [4], [6], [10], [11], [12], [14], [21], etc.). Recently, a lot of attention has been paid to the following Volterra-Lotka type two species competition systems with nonlocal dispersal,

$$\begin{cases} u_t = \nu_1 [\int_D \kappa(y - x)u(t, y)dy - u(t, x)] + u(a_1(t, x) - b_1(t, x)u - c_1(t, x)v), & x \in \bar{D} \\ v_t = \nu_2 [\int_D \kappa(y - x)v(t, y)dy - v(t, x)] + v(a_2(t, x) - b_2(t, x)u - c_2(t, x)v), & x \in \bar{D}, \end{cases} \quad (1.3)$$

$$\begin{cases} u_t = \nu_1 \int_D \kappa(y - x)[u(t, y) - u(t, x)]dy + u(a_1(t, x) - b_1(t, x)u - c_1(t, x)v), & x \in \bar{D} \\ v_t = \nu_2 \int_D \kappa(y - x)[v(t, y) - v(t, x)]dy + v(a_2(t, x) - b_2(t, x)u - c_2(t, x)v), & x \in \bar{D}, \end{cases} \quad (1.4)$$

and

$$\begin{cases} u_t = \nu_1 \int_{\mathbb{R}^N} \kappa(y - x)[u(t, y) - u(t, x)]dy + u(a_1(t, x) - b_1(t, x)u - c_1(t, x)v), & x \in \mathbb{R}^N \\ v_t = \nu_2 \int_{\mathbb{R}^N} \kappa(y - x)[v(t, y) - v(t, x)]dy + v(a_2(t, x) - b_2(t, x)u - c_2(t, x)v), & x \in \mathbb{R}^N \\ u(t, x + p_k \mathbf{e}_k) = u(t, x), & v(t, x + p_k \mathbf{e}_k) = v(t, x), \quad k = 1, 2, \dots, N, \end{cases} \quad (1.5)$$

where $\kappa(\cdot)$ is a nonnegative symmetric smooth function with support $B(0, r) = \{x \in \mathbb{R}^N \mid \|x\| < r\}$ for some $r > 0$, $\int_{\mathbb{R}^N} \kappa(z)dz = 1$, and in (1.5), $a_i(t, x + p_k \mathbf{e}_k) = a_i(t, x)$, $b_i(t, x + p_k \mathbf{e}_k) = b_i(t, x)$,

and $c_i(t, x + p_k \mathbf{e}_k) = c_i(t, x)$ for $i = 1, 2$, $k = 1, 2, \dots, N$, where $p_k > 0$ (see [2], [3], [17], [26], etc.). We point out that the works [23] and [24] considered two species competition systems which involves both random and nonlocal dispersals.

Thanks to the relation between the nonlocal dispersal operator in (1.3) (resp. (1.4), (1.5)) and the random dispersal operator in (1.2) with Dirichlet boundary condition (resp. Neumann boundary condition, periodic boundary condition) (see [7], [8], [31]), (1.3) (resp. (1.4), (1.5)) is referred to as a Volterra-Lotka type competition system with nonlocal dispersal and Dirichlet type boundary condition (resp. Neumann type boundary condition, periodic boundary condition).

Coexistence and extinction dynamics is among the central problems investigated for (1.2), (1.3), (1.4), and (1.5). Roughly, we say that (1.2) with time periodic coefficients has a *coexistence state* if it has a time periodic solution $(u^{**}(t, x), v^{**}(t, x))$ with $u^{**}(t, x), v^{**}(t, x) > 0$ for $x \in D$. We say that the *species v is eventually driven to extinction* if $\lim_{t \rightarrow \infty} v(t, x) = 0$ holds for every solution $(u(t, x), v(t, x))$ of (1.2) with $u(t, x) > 0$, $v(t, x) > 0$. Note that, by the regularity of solutions for parabolic equations, a coexistence state $(u^{**}(t, x), v^{**}(t, x))$ of (1.2) (if exists) is continuous in x . We say that (1.3), (resp. (1.4), (1.5)) with time periodic coefficients has a *coexistence state* if it has a time periodic solution $(u^{**}(t, x), v^{**}(t, x))$ with $u^{**}(t, x), v^{**}(t, x) > 0$ for $x \in \bar{D}$ and being continuous in $x \in \bar{D}$ (in the case (1.5), $x \in \mathbb{R}^N$). We say that the *species v is eventually driven to extinction* if $\lim_{t \rightarrow \infty} v(t, x) = 0$ holds for every solution $(u(t, x), v(t, x))$ of (1.3), (resp. (1.4), (1.5)) with $u(t, x) > 0$, $v(t, x) > 0$.

There are many studies on the coexistence and extinction dynamics of (1.2) with a_i , b_i , and c_i being time independent or periodic (see [1], [9], [13], [18], [19], [33], [34], etc.). In [17], the authors studied coexistence and extinction dynamics of (1.3), (1.4), and (1.5) with a_i , b_i , and c_i being independent of t . Consider the following spectral problems,

$$\int_D \kappa(y) u(y) dy - u(x) = \lambda u(x), \quad x \in \bar{D}, \quad u \in C(\bar{D}), \quad (1.6)$$

$$\int_D \kappa(y - x) [u(y) - u(x)] dy = \lambda u(x), \quad x \in \bar{D}, \quad u \in C(\bar{D}), \quad (1.7)$$

and

$$\int_{\mathbb{R}^N} \kappa(y - x) [u(y) - u(x)] dy = \lambda u(x), \quad x \in \bar{\mathbb{R}}^N, \quad u \in C(\mathbb{R}^N). \quad (1.8)$$

Let λ_0^D , λ_0^N , λ_0^P be the principal eigenvalues of (1.6), (1.7), and (1.8), respectively. See section 2 for the existence of λ_0^D , λ_0^N and λ_0^P . It should be noted that $\lambda_0^D < 0$, $\lambda_0^N = 0$, and $\lambda_0^P = 0$.

Set

$$\begin{cases} a_{iL(M)} = \inf_{t \in \mathbb{R}, x \in \bar{D}} (\sup_{t \in \mathbb{R}, x \in \bar{D}}) a_i(t, x) \\ b_{iL(M)} = \inf_{t \in \mathbb{R}, x \in \bar{D}} (\sup_{t \in \mathbb{R}, x \in \bar{D}}) b_i(t, x) \\ c_{iL(M)} = \inf_{t \in \mathbb{R}, x \in \bar{D}} (\sup_{t \in \mathbb{R}, x \in \bar{D}}) c_i(t, x) \end{cases} \quad (1.9)$$

for $i = 1, 2$, where $D = \mathbb{R}^N$ in the case of (1.5). The following two theorems on the coexistence and extinction of time independent competing systems with nonlocal dispersal are proved in [17].

Theorem A' (Coexistence states) *Consider (1.3) with a_i, b_i, c_i ($i = 1, 2$) being independent of t and assume that $a_{iL} > -\nu_i \lambda_0^D$ for $i = 1, 2$.*

- (1) *If $a_{1L} > -\nu_1 \lambda_0^D + \frac{c_{1M} a_{2M}}{c_{2L}}$ and $a_{2L} > -\nu_2 \lambda_0^D + \frac{b_{2M} a_{1M}}{b_{1L}}$, then (1.3) has at least one coexistence state $(u^{**}(x), v^{**}(x))$.*
- (2) *If $\nu_1 = \nu_2$, $a_1(x) = a_2(x)$, and $b_1(x) > b_2(x)$, $c_1(x) < c_2(x)$ for $x \in \bar{D}$, then (1.3) has at least one coexistence state $(u^{**}(x), v^{**}(x))$.*
- (3) *If $\nu_1 = \nu_2$, $a_1(x) = a_2(x)$ for $x \in \bar{D}$, and b_i, c_i ($i = 1, 2$) are constant functions with $b_1 > b_2$ and $c_1 < c_2$, then (1.3) has a unique globally stable coexistence state $(u^{**}(x), v^{**}(x))$.*

Theorem B' (Extinction) *Consider (1.3) with a_i, b_i, c_i ($i = 1, 2$) being independent of t and assume that $a_{iL} > -\nu_i \lambda_0^D$ for $i = 1, 2$.*

- (1) *If $a_{1L} > \frac{c_{1M} a_{2M}}{c_{2L}}$, $a_{2M} \leq \frac{a_{1L} b_{2L}}{b_{1M}}$, $\nu_1 = \nu_2$, and $a_{1L} \geq a_{2M}$, then species v is eventually driven to extinction.*
- (2) *If $a_{1M} \leq \frac{c_{1L} a_{2L}}{c_{2M}}$, $a_{2L} > \frac{a_{1M} b_{2M}}{b_{1L}}$, $\nu_1 = \nu_2$, and $a_{1M} \leq a_{2L}$, then species u is eventually driven to extinction.*
- (3) *If $\nu_1 < \nu_2$, and $a_1(x) = a_2(x)$, $b_1(x) = b_2(x)$, $c_1(x) = c_2(x)$, then species v is eventually driven to extinction.*

Similar results to Theorem A' and Theorem B' have been proved in [17] for (1.4) and (1.5) (see Theorems C, D, E, and F in [17]).

Up to our knowledge, there is little study on the coexistence and extinction dynamics of (1.3), (1.4) and (1.5) with time periodic coefficients. The objective of this paper is to study the coexistence and extinction dynamics of (1.3), (1.4) and (1.5) with time periodic coefficients. Throughout the rest of this paper, $D = \mathbb{R}^N$ when (1.5) is considered. We recall that the following results are proved in [30].

- Consider (1.3) (resp. (1.4), (1.5)) and assume $a_{iL} > -\nu_i \lambda_0^D$ (resp. $a_{iL} > -\nu_i \lambda_0^N$, $a_{iL} > -\nu_i \lambda_0^P$) for $i = 1, 2$. Then (1.3) (resp. (1.4), (1.5)) has a semitrivial time periodic solution $(u^*(t, \cdot), 0) \in (C(\bar{D}, \mathbb{R}) \setminus \{0\}) \times C(\bar{D}, \mathbb{R})$ which is globally semi-stable in the sense that for any $u_0 \in C(\bar{D}, \mathbb{R})$ with $u_0 \geq 0$ and $u_0 \not\equiv 0$, $(u(t, \cdot; u_0, 0), v(t, \cdot; u_0, 0)) - (u^*(t, \cdot), 0) \rightarrow (0, 0)$ as $t \rightarrow \infty$ (see Proposition 2.5).
- Consider (1.3) (resp. (1.4), (1.5)) and assume $a_{iL} > -\nu_i \lambda_0^D$ (resp. $a_{iL} > -\nu_i \lambda_0^N$, $a_{iL} > -\nu_i \lambda_0^P$) for $i = 1, 2$. Then (1.3) (resp. (1.4), (1.5)) has a semitrivial time periodic solution

$(0, v^*(t, \cdot)) \in C(\bar{D}, \mathbb{R}) \times (C(\bar{D}, \mathbb{R}) \setminus \{0\})$ which is globally semi-stable in the sense that for any $v_0 \in C(\bar{D}, \mathbb{R})$ with $v_0 \geq 0$ and $v_0 \not\equiv 0$, $(u(t, \cdot; 0, v_0), v(t, \cdot; 0, v_0)) - (0, v^*(t, \cdot)) \rightarrow (0, 0)$ as $t \rightarrow \infty$ (see Proposition 2.5).

We will prove the following theorems in this paper.

Theorem A. (Coexistence states) *Consider (1.3) (resp. (1.4), (1.5)) and assume that a_i, b_i and c_i are periodic in t with period T and $a_{iL} > -\nu_i \lambda_0^D$ (resp. $a_{iL} > -\nu_i \lambda_0^N, a_{iL} > -\nu_i \lambda_0^P$) for $i = 1, 2$.*

- (1) *If $a_{1L} > -\nu_1 \lambda_0^D + \frac{c_{1M} a_{2M}}{c_{2L}}$ and $a_{2L} > -\nu_2 \lambda_0^D + \frac{b_{2M} a_{1M}}{b_{1L}}$ (resp. $a_{1L} > -\nu_1 \lambda_0^N + \frac{c_{1M} a_{2M}}{c_{2L}}$ and $a_{2L} > -\nu_2 \lambda_0^N + \frac{b_{2M} a_{1M}}{b_{1L}}$, $a_{1L} > -\nu_1 \lambda_0^P + \frac{c_{1M} a_{2M}}{c_{2L}}$ and $a_{2L} > -\nu_2 \lambda_0^P + \frac{b_{2M} a_{1M}}{b_{1L}}$), then (1.3) (resp. (1.4), (1.5)) has at least one coexistence state $(u^{**}(t, x), v^{**}(t, x)) = (u^{**}(t + T, x), v^{**}(t + T, x))$.*
- (2) *If $\nu_1 = \nu_2$, $a_1(t, x) = a_2(t, x)$, and $\inf_{x \in \bar{D}} b_1(t, x) > \sup_{x \in \bar{D}} b_2(t, x)$, $\sup_{x \in \bar{D}} c_1(t, x) < \inf_{x \in \bar{D}} c_2(t, x)$ for $t \in \mathbb{R}$, then (1.3) (resp. (1.4), (1.5)) has at least one coexistence state $(u^{**}(t, x), v^{**}(t, x)) = (u^{**}(t + T, x), v^{**}(t + T, x))$.*
- (3) *If $\nu_1 = \nu_2$, $a_1(t, x) = a_2(t, x)$ for $x \in \bar{D}$ and $t \in \mathbb{R}$, and b_i, c_i ($i = 1, 2$) are constant functions with $b_1 > b_2$ and $c_1 < c_2$, then (1.3) (resp. (1.4), (1.5)) has a unique globally stable coexistence state $(u^{**}(t, x), v^{**}(t, x)) = (u^{**}(t + T, x), v^{**}(t + T, x))$.*

Theorem B. (Extinction) *Consider (1.3) (resp. (1.4), (1.5)) and assume that a_i, b_i and c_i are periodic in t with period T and $a_{iL} > -\nu_i \lambda_0^D$ (resp. $a_{iL} > -\nu_i \lambda_0^N, a_{iL} > -\nu_i \lambda_0^P$) for $i = 1, 2$.*

- (1) *If $a_{1L} > \frac{c_{1M} a_{2M}}{c_{2L}}$, $a_{2M} \leq \frac{a_{1L} b_{2L}}{b_{1M}}$, $\nu_1 = \nu_2$, and $a_{1L} \geq a_{2M}$, then $(u^*(t, x), 0)$ is globally stable and hence species v is eventually driven to extinction.*
- (2) *If $a_{1M} \leq \frac{c_{1L} a_{2L}}{c_{2M}}$, $a_{2L} > \frac{a_{1M} b_{2M}}{b_{1L}}$, $\nu_1 = \nu_2$, and $a_{1M} \leq a_{2L}$, then $(0, v^*(t, x))$ is globally stable and hence species u is eventually driven to extinction.*

The above results extend Theorem A' and Theorem B' for time independent Volterra-Lotka type two species competition system with nonlocal dispersal to time periodic ones. They also extend the existing results on coexistence and extinction dynamics for time periodic Volterra-Lotka type two species competition system with random dispersal to such systems with nonlocal dispersal.

Note that ecologically, Theorem B' (3) indicates that in time independent and spatially inhomogeneous media, the species with slower diffusion is selected for. Such scenario may not be true for two species competition systems with random dispersal in time periodic and spatially inhomogeneous media (see [22]). We conjecture that the scenario may also not be true for two species competition systems with nonlocal dispersal in time periodic and spatially inhomogeneous media.

It should be pointed out that several difficulties arise in dealing with (1.3) (resp. (1.4), (1.5)) when following the general approach for (1.2). This is due to the fact that the solution operator of (1.3) (resp. (1.4), (1.5)) lacks smoothness and compactness in suitable phase spaces. The main tools employed in the study of (1.3), (1.4), and (1.5) include principal spectral theory for nonlocal dispersal operators with time periodic dependence, comparison principle for (1.3), (1.4), and (1.5), and sub- and super-solutions.

The rest of this paper is organized as follows. In section 2, we present some preliminary materials for the use in later sections. Sections 3 and 4 are devoted to the proofs of Theorems A and B, respectively.

2 Preliminary

In this section, we present some preliminary materials for the use in later sections, including principal spectrum theory for nonlocal dispersal operators with time periodic dependence, semitrivial time periodic solutions of time periodic Volterra-Lotka type two species competition systems with nonlocal dispersal, and comparison principle for Volterra-Lotka type two species competition systems with nonlocal dispersal.

2.1 Principal spectrum theory of nonlocal dispersal operators with time periodic dependence

In this subsection, we present some principal spectrum theory for nonlocal dispersal operators with time periodic dependence.

Let

$$X_1 = X_2 = C(\bar{D}, \mathbb{R})$$

with norm $\|u\|_{X_i} = \sup_{x \in \bar{D}} |u(x)|$ ($i = 1, 2$),

$$X_3 = \{u \in C(\mathbb{R}^N, \mathbb{R}) \mid u(x + p_j \mathbf{e}_j) = u(x)\}$$

with norm $\|u\|_{X_3} = \sup_{x \in \mathbb{R}^N} |u(x)|$, and

$$X_i^+ = \{u \in X_i \mid u \geq 0\}, \quad i = 1, 2, 3,$$

$$X_i^{++} = \begin{cases} \{u \in X_i^+ \mid u(x) > 0 \quad \forall x \in \bar{D}\}, & i = 1, 2 \\ \{u \in X_i^+ \mid u(x) > 0 \quad \forall x \in \mathbb{R}^N\}, & i = 3. \end{cases}$$

For given $\nu_i > 0$ and $l_i(\cdot) \in X_i$ ($i = 1, 2, 3$), let $L_i^0(\nu_i, l_i) : \mathcal{D}(L_i^0(\nu_i, l_i)) \subset X_i \rightarrow X_i$ be defined as follows,

$$(L_1^0(\nu_1, l_1)u)(x) = \nu_1 \left[\int_D \kappa(y - x)u(y)dy - u(x) \right] + l_1(x)u(x),$$

$$(L_2^0(\nu_2, l_2)u)(x) = \nu_2 \left[\int_D \kappa(y - x)(u(y) - u(x))dy \right] + l_2(x)u(x),$$

and

$$(L_3^0(\nu_3, l_3)u)(x) = \nu_3 \left[\int_{\mathbb{R}^N} \kappa(y-x)u(y)dy - u(x) \right] + l_3(x)u(x).$$

Let

$$\mathcal{X}_1 = \mathcal{X}_2 = \{u \in C(\mathbb{R} \times \bar{D}, \mathbb{R}) \mid u(t+T, x) = u(t, x)\}$$

with norm $\|u\|_{\mathcal{X}_i} = \sup_{t \in \mathbb{R}, x \in \bar{D}} |u(t, x)|$ ($i = 1, 2$),

$$\mathcal{X}_3 = \{u \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) \mid u(t+T, x) = u(t, x + p_i \mathbf{e}_i) = u(t, x)\}$$

with norm $\|u\|_{\mathcal{X}_3} = \sup_{t \in \mathbb{R}, x \in \mathbb{R}^N} |u(t, x)|$, and

$$\mathcal{X}_i^+ = \{u \in \mathcal{X}_i \mid u \geq 0\}, \quad i = 1, 2, 3.$$

For given $\nu_i > 0$ and $l_i \in \mathcal{X}_i$ ($i = 1, 2, 3$), let $L_i(\nu_i, l_i) : \mathcal{D}(L_i(\nu_i, l_i)) \subset \mathcal{X}_i \rightarrow \mathcal{X}_i$ be defined as follows,

$$(L_1(\nu_1, l_1)u)(t, x) = -u_t(t, x) + \nu_1 \left[\int_D \kappa(y-x)u(t, y)dy - u(t, x) \right] + l_1(t, x)u(t, x),$$

$$(L_2(\nu_2, l_2)u)(t, x) = -u_t(t, x) + \nu_2 \left[\int_D \kappa(y-x)(u(t, y) - u(t, x))dy \right] + l_2(t, x)u(t, x),$$

and

$$(L_3(\nu_3, l_3)u)(t, x) = -u_t(t, x) + \nu_3 \left[\int_{\mathbb{R}^N} \kappa(y-x)u(t, y)dy - u(t, x) \right] + l_3(t, x)u(t, x).$$

Definition 2.1. (1) Let

$$\lambda_i^0(\nu_i, l_i) = \sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(L_i^0(\nu_i, l_i))\}$$

for $i = 1, 2, 3$, where $l_i \in \mathcal{X}_i$. $\lambda_i^0(\nu_i, l_i)$ is called the principal spectrum point of $L_i^0(\nu_i, l_i)$ ($i = 1, 2, 3$). If $\lambda_i^0(\nu_i, l_i)$ is an isolated eigenvalue of $L_i^0(\nu_i, l_i)$ with a positive eigenfunction ϕ (i.e. $\phi \in \mathcal{X}_i^+$), then $\lambda_i^0(\nu_i, l_i)$ is called the principal eigenvalue of $L_i^0(\nu_i, l_i)$ or it is said that $L_i^0(\nu_i, l_i)$ has a principal eigenvalue ($i = 1, 2, 3$).

(2) Let

$$\lambda_i(\nu_i, l_i) = \sup\{\operatorname{Re} \lambda \mid \lambda \in \sigma(L_i(\nu_i, l_i))\}$$

for $i = 1, 2, 3$, where $l_i \in \mathcal{X}_i$. $\lambda_i(\nu_i, l_i)$ is called the principal spectrum point of $L_i(\nu_i, l_i)$ ($i = 1, 2, 3$). If $\lambda_i(\nu_i, l_i)$ is an isolated eigenvalue of $L_i(\nu_i, l_i)$ with a positive eigenfunction ϕ (i.e. $\phi \in \mathcal{X}_i^+$), then $\lambda_i(\nu_i, l_i)$ is called the principal eigenvalue of $L_i(\nu_i, l_i)$ or it is said that $L_i(\nu_i, l_i)$ has a principal eigenvalue ($i = 1, 2, 3$).

Remark 2.1. For given $1 \leq i \leq 3$ and $l_i(\cdot, \cdot) \in \mathcal{X}_i$, if $l_i(t, x) = l_i(x)$, i.e., $l_i(t, x)$ is independent of t , then $\lambda_i(\nu_i, l_i) = \lambda_i^0(\nu_i, l_i)$.

For given $1 \leq i \leq 3$ and $l_i \in \mathcal{X}_i$, let $\hat{l}_i(x)$ be the time average of $l_i(t, x)$ ($i = 1, 2, 3$), that is,

$$\hat{l}_i(x) = \frac{1}{T} \int_0^T l_i(t, x) dt, \quad T > 0 \quad (2.1)$$

and

$$m_i(x) = \begin{cases} -\nu_i & \text{for } i = 1, 3 \\ -\nu_2 \int_D \kappa(y - x) dy & \text{for } i = 2. \end{cases} \quad (2.2)$$

Let

$$D_i = \begin{cases} \bar{D} & \text{for } i = 1, 2 \\ [0, p_1] \times [0, p_2] \times \cdots \times [0, p_N] & \text{for } i = 3. \end{cases} \quad (2.3)$$

Proposition 2.1. *Let $\nu_i > 0$ and $l_i \in \mathcal{X}_i$ ($1 \leq i \leq 3$) be given. If $\lambda \in \mathbb{R}$ is an eigenvalue of $L_i(\nu_i, l_i)$ with a positive eigenfunction $\phi(t, x)$, then λ is the principal eigenvalue of $L_i(\nu_i, l_i)$. Moreover, $\lambda = \lambda_i(\nu_i, l_i) > \max_{x \in D_i} (m_i(x) + \hat{l}_i(x))$. Conversely, if $\lambda_i(\nu_i, l_i) > \max_{x \in D_i} (m_i(x) + \hat{l}_i(x))$, then $\lambda_i(\nu_i, l_i)$ is the principal eigenvalue of $L_i(\nu_i, l_i)$. Hence, $\lambda_i(\nu_i, l_i)$ is the principal eigenvalue of $L_i(\nu_i, l_i)$ iff $\lambda_i(\nu_i, l_i) > \max_{x \in D_i} (m_i(x) + \hat{l}_i(x))$.*

Proof. See [30, Theorem A]. □

Proposition 2.2. *Let $\nu_i > 0$ and $l_i \in \mathcal{X}_i$ ($1 \leq i \leq 3$) be given. The principal eigenvalue of $L_i(\nu_i, l_i)$ exists if $m_i(x) + \hat{l}_i(x)$ is C^N , there is some $x_0 \in \text{Int}(D_i)$ in the case $i = 1, 2$ and $x_0 \in D_i$ in the case $i = 3$ satisfying that $m_i(x_0) + \hat{l}_i(x_0) = \max_{x \in D_i} (m_i(x) + \hat{l}_i(x))$, and the partial derivatives of $m_i(x) + \hat{l}_i(x)$ up to order $N - 1$ at x_0 are zero.*

Proof. See [30, Theorem B]. □

Proposition 2.3. (1) *For given $1 \leq i \leq 3$, $\nu_i > 0$, and $l_i, \tilde{l}_i \in \mathcal{X}_i$ with $l_i(t, x) \leq \tilde{l}_i(t, x)$,*

$$\lambda_i(\nu_i, l_i) \leq \lambda_i(\nu_i, \tilde{l}_i).$$

(2) *For given $1 \leq i \leq 3$, $\nu_i > 0$, $l_i \in \mathcal{X}_i$, and any constant $a \in \mathbb{R}$,*

$$\lambda_i(\nu_i, l_i + a) = \lambda_i(\nu_i, l_i) + a.$$

Proof. (1) It follows from [30, Propositions 3.2 and 3.10].

(2) It follows directly from the definition. □

Proposition 2.4. *For given $1 \leq i \leq 3$, $\nu_i > 0$, and $l_i \in \mathcal{X}_i$, if there is $\phi_i \in \mathcal{X}_i^+ \setminus \{0\}$ such that*

$$L_i(\nu_i, l_i)\phi_i = 0,$$

then $\lambda_i(\nu_i, l_i) = 0$.

Proof. It follows from [30, Propositions 3.2 and 3.10]. □

We remark that

$$\lambda_0^D = \lambda_1(1, 0) < 0, \quad \lambda_0^N = \lambda_2(1, 0) = 0, \quad \lambda_0^P = \lambda_3(1, 0) = 0,$$

and

$$\lambda_1^0(\nu_1, 0) = \nu_1 \lambda_0^D, \quad \lambda_2^0(\nu_2, 0) = \nu_2 \lambda_0^N, \quad \lambda_3^0(\nu_3, 0) = \nu_3 \lambda_0^P.$$

2.2 Semitrivial time periodic solutions

In this section, we recall the existence and stability of semitrivial time periodic solutions of (1.3), (1.4), and (1.5).

First of all, let X_i, X_i^+, X_i^{++} ($1 \leq i \leq 3$) be as in subsection 2.1. Semigroup theory (see [16], [29]) guarantees for $(u_0, v_0) \in X_1 \times X_1$ (resp. $(u_0, v_0) \in X_2 \times X_2, (u_0, v_0) \in X_3 \times X_3$) that (1.3) (resp. (1.4), (1.5)) has a unique (local) solution $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0))$ with $(u(0, \cdot; u_0, v_0), v(0, \cdot; u_0, v_0)) = (u_0, v_0)$. Moreover, if $(u_0, v_0) \in X_i \times \{0\}$ ($\{0\} \times X_i$), then $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) \in X_i \times \{0\}$ ($\{0\} \times X_i$).

Proposition 2.5. *If $a_{iL} > -\nu_i \lambda_0^D$ for $i = 1, 2$ (resp. $a_{iL} > -\nu_i \lambda_0^N$ for $i = 1, 2$, $a_{iL} > -\nu_i \lambda_0^P$ for $i = 1, 2$), then (1.3) (resp. (1.4), (1.5)) has two semitrivial time periodic solutions $(u^*(t, x), 0)$ and $(0, v^*(t, x))$ with $u^*(t, \cdot), v^*(t, \cdot) \in X_1^{++}$ (resp. $u^*(t, \cdot), v^*(t, \cdot) \in X_2^{++}, u^*(t, \cdot), v^*(t, \cdot) \in X_3^{++}$). Moreover, for any $(u_0, v_0) \in (X_1^+ \setminus \{0\}) \times \{0\}$ (resp. $(u_0, v_0) \in (X_2^+ \setminus \{0\}) \times \{0\}, (u_0, v_0) \in (X_3^+ \setminus \{0\}) \times \{0\}$,*

$$(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) - (u^*(t, \cdot), 0) \rightarrow (0, 0) \quad \text{as } t \rightarrow \infty,$$

and for any $(u_0, v_0) \in \{0\} \times (X_1^+ \setminus \{0\})$ (resp. $(u_0, v_0) \in \{0\} \times (X_2^+ \setminus \{0\}), (u_0, v_0) \in \{0\} \times (X_3^+ \setminus \{0\})$,

$$(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) - (0, v^*(t, \cdot)) \rightarrow (0, 0) \quad \text{as } t \rightarrow \infty,$$

where $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0))$ is the solution of (1.3) (resp. (1.4), (1.5)) with initial (u_0, v_0) .

Proof. We give a proof for (1.3). It can be proved similarly for (1.4) and (1.5).

First, we note that

$$\lambda_1(\nu_1, a_1) \geq \lambda_1(\nu_1, a_{1L}) = \nu_1 \lambda_1(1, 0) + a_{1L} = \nu_1 \lambda_0^D + a_{1L}.$$

Hence $\lambda_1(\nu_1, a_1) > 0$. Then by [30, Theorem E], (1.3) has a semitrivial periodic solution $(u^*(t, x), 0)$ satisfying that $u^*(t, \cdot) \in X_1^{++}$ and for any $(u_0, 0) \in (X_1^+ \setminus \{0\}) \times \{0\}$,

$$(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) - (u^*(t, \cdot), 0) \rightarrow (0, 0)$$

as $t \rightarrow \infty$, where $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0))$ is the solution of (1.3) with initial (u_0, v_0) .

Similarly, (1.3) has a semitrivial periodic solution $(0, v^*(t, x))$ satisfying that $v^*(t, \cdot) \in X_1^{++}$ and for any $(u_0, 0) \in \{0\} \times (X_1^+ \setminus \{0\})$,

$$(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) - (0, v^*(t, \cdot)) \rightarrow (0, 0)$$

as $t \rightarrow \infty$, where $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0))$ is the solution of (1.3) with initial (u_0, v_0) . \square

2.3 Comparison principle

In this subsection, we recall a comparison for solutions of (1.3), (1.4), and (1.5).

For $u_1, u_2 \in X_i$ ($1 \leq i \leq 3$), we define

$$u_1 \leq u_2 \text{ } (u_1 \geq u_2) \text{ if } u_2 - u_1 \in X_i^+ \text{ } (u_1 - u_2 \in X_i^+),$$

and

$$u_1 \ll u_2 \text{ } (u_1 \gg u_2) \text{ if } u_2 - u_1 \in X_i^{++} \text{ } (u_1 - u_2 \in X_i^{++}).$$

Define the following orderings in $X_i \times X_i$:

$$(u_1, v_1) \leq_1 (\ll_1)(u_2, v_2) \text{ if } u_1 \leq (\ll)u_2, v_1 \leq (\ll)v_2, \quad (2.4)$$

$$(u_1, v_1) \leq_2 (\ll_2)(u_2, v_2) \text{ if } u_1 \leq (\ll)u_2, v_1 \geq (\gg)v_2. \quad (2.5)$$

Observe that \leq_1 is the usual order and \leq_2 is called the *competitive order* in the literature.

Let $\tau > 0$ and $(u, v) \in C([0, \tau] \times \bar{D}, \mathbb{R}^2)$ with $(u(t, \cdot), v(t, \cdot)) \in X_1^+ \times X_1^+$. Then (u, v) is called a *super-solution* (*sub-solution*) of (1.3) on $[0, \tau]$ if

$$\begin{cases} u_t \geq (\leq) \nu_1 [\int_D k(y-x)u(t, y)dy - u(t, x)] + u[a_1(t, x) - b_1(t, x)u - c_1(t, x)v], & x \in \bar{D}, \\ v_t \leq (\geq) \nu_2 [\int_D k(y-x)v(t, y)dy - v(t, x)] + v[a_2(t, x) - b_2(t, x)u - c_2(t, x)v], & x \in \bar{D}, \end{cases}$$

for $t \in (0, \tau)$. Super-solutions and sub-solutions of (1.4) and (1.5) are defined similarly.

Proposition 2.6. (1) Consider (1.3) (resp. (1.4), (1.5)). For given $(u_0, v_0) \in X_1 \times X_1$ (resp. $(u_0, v_0) \in X_2 \times X_2, (u_0, v_0) \in X_3 \times X_3$), if $(0, 0) \leq_1 (u_0, v_0)$, then $(0, 0) \leq_1 (u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0))$ for all $t > 0$ at which $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0))$ exists, where $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0))$ is the solution of (1.3) (resp. (1.4), (1.5)) with initial (u_0, v_0) .

(2) If $(0, 0) \leq_1 (u_i(t, \cdot), v_i(t, \cdot))$ for $i = 1, 2$, $(u_1(0, \cdot), v_1(0, \cdot)) \leq_2 (u_2(0, \cdot), v_2(0, \cdot))$, and $(u_1(t, x), v_1(t, x))$ is a sub-solution and $(u_2(t, x), v_2(t, x))$ is a super-solution of (1.3) (resp. (1.4), (1.5)) on $[0, \tau]$, then $(u_1(t, \cdot), v_1(t, \cdot)) \leq_2 (u_2(t, \cdot), v_2(t, \cdot))$ for $t \in (0, \tau)$.

(3) Consider (1.3) (resp. (1.4), (1.5)). For given $(u_i, v_i) \in X_1 \times X_1$ (resp. $(u_i, v_i) \in X_2 \times X_2, (u_i, v_i) \in X_3 \times X_3$) ($i = 1, 2$), if $(0, 0) \leq_1 (u_i, v_i)$ for $i = 1, 2$ and $(u_1, v_1) \leq_2 (u_2, v_2)$, then

$$(u(t, \cdot; u_1, v_1), v(t, \cdot; u_1, v_1)) \leq_2 (u(t, \cdot; u_2, v_2), v(t, \cdot; u_2, v_2))$$

for all $t > 0$ at which both $(u(t, \cdot; u_1, v_1), v(t, \cdot; u_1, v_1))$ and $(u(t, \cdot; u_2, v_2), v(t, \cdot; u_2, v_2))$ exist, where $(u(t, \cdot; u_i, v_i), v(t, \cdot; u_i, v_i))$ is the solution of (1.3) (resp. (1.4), (1.5)) with initial (u_i, v_i) .

(4) Let $(u_0, v_0) \in X_i^+ \times X_i^+$ ($i = 1, 2, 3$), then $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0))$ exists for all $t > 0$.

Proof. It follows from the arguments of [17, Proposition 3.1]. \square

3 Existence, Uniqueness, and Stability of Coexistence States

In this section, we investigate the existence, uniqueness, and stability of coexistence states of (1.3), (1.4), and (1.5), and prove Theorem A. We first prove the following theorem.

Theorem 3.1. *Assume that $\frac{\inf_{t \in \mathbb{R}} b_1(t, x)}{\sup_{t \in \mathbb{R}} b_2(t, x)} > \frac{\sup_{t \in \mathbb{R}} c_1(t, x)}{\inf_{t \in \mathbb{R}} c_2(t, x)}$ for each $x \in \bar{D}$. If $(u^{**}(t, x), v^{**}(t, x))$ is a measurable coexistence state of (1.3) (resp. (1.4), (1.5)), then $(u^{**}(t, x), v^{**}(t, x))$ is continuous in $x \in \bar{D}$.*

Observe that, by Theorem 3.1, to prove Theorem A, it suffices to prove the existence of a measurable coexistence state. To prove Theorem 3.1, we first prove a lemma.

Lemma 3.1. *Consider*

$$\begin{cases} u_t = u(a_1(t) - b_1(t)u - c_1(t)v) + d_1(t) \\ v_t = v(a_2(t) - b_2(t)u - c_2(t)v) + d_2(t), \end{cases} \quad (3.1)$$

where $b_i(\cdot)$, $c_i(\cdot)$, and $d_i(\cdot)$ ($i = 1, 2$) are positive continuous periodic functions with period T . Assume

$$\frac{b_{1L}}{b_{2M}} > \frac{c_{1M}}{c_{2L}}.$$

Then (3.1) has a unique time periodic positive solution.

Proof. It follows from Theorem 2.3.1 in [32]. In the following, we provide the idea of proof.

First of all, there is a unique time periodic stable solution $u^*(t)$ of

$$\dot{u} = u(a_1(t) - b_1(t)u) + d_1(t)$$

and there is a unique time periodic stable solution $v^*(t)$ of

$$\dot{v} = v(a_2(t) - c_2(t)v) + d_2(t)$$

(see [32, Proposition 2.2]). Then, by Proposition 2.6,

$$(0, v^*(0)) \ll_2 (u(T; u^*(0), 0), v(T; u^*(0), 0)) \ll_2 (u^*(0), 0).$$

This implies that

$$(u((n+1)T; u^*(0), 0), v((n+1)T; u^*(0), 0)) \ll_2 (u(nT; u^*(0), 0), v(nT; u^*(0), 0)) \ll_2 (u^*(0), 0)$$

and

$$(0, v^*(0)) \ll_2 (u((n+1)T; u^*(0), 0), v((n+1)T; u^*(0), 0)) \ll_2 (u(nT; u^*(0), 0), v(nT; u^*(0), 0))$$

Hence $\lim_{n \rightarrow \infty} (u(nT; u^*(0), 0), v(nT; u^*(0), 0))$ exists. Let

$$(u_0^+, v_0^+) = \lim_{n \rightarrow \infty} (u(nT; u^*(0), 0), v(nT; u^*(0), 0)).$$

We have that

$$(u^+(t), v^+(t)) := (u(t; u_0^+, v_0^+), v(t; u_0^+, v_0^+))$$

is a periodic solution of (3.1).

Next, by comparison principle for competition systems of ODEs, for any $(u_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $(u_0, v_0) \geq_2 (u^+(0), v^+(0))$,

$$(u^+(t), v^+(t)) \leq_2 (u(t; u_0, v_0), v(t; u_0, v_0))$$

for all $t \geq 0$. Note that $u(t; u_0, v_0)$ satisfies

$$\dot{u} = u(a_1(t) - b_1(t)u - c_1(t)v(t; u_0, v_0)) + d_1(t) < u(a_1(t) - b_1(t)u) + d_1(t).$$

Then there is $N^* \geq 1$ such that

$$u(nT; u_0, v_0) \leq u^*(0)$$

for $n \geq N^*$. This implies that

$$(u(nT; u_0, v_0), v(nT; u_0, v_0)) \leq_2 (u^*(0), 0)$$

for $n \geq N^*$. It can then be proved that

$$\lim_{t \rightarrow \infty} [(u(t; u_0, v_0), v(t; u_0, v_0)) - (u^+(t), v^+(t))] = 0.$$

Similarly, we can prove the existence of the limit

$$(u_0^-, v_0^-) = \lim_{n \rightarrow \infty} (u(nT; 0, v^*(0)), v(nT; 0, v^*(0)))$$

and that $(u^-(t), v^-(t)) := (u(t; u_0^-, v_0^-), v(t; u_0^-, v_0^-))$ is a periodic solution of (3.1) satisfying that

$$\lim_{t \rightarrow \infty} [(u(t; u_0, v_0), v(t; u_0, v_0)) - (u^-(t), v^-(t))] = 0$$

for any $(u_0, v_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $(u_0, v_0) \leq_2 (u^-(0), v^-(0))$.

It now suffices to prove that

$$(u^-(t), v^-(t)) \equiv (u^+(t), v^+(t)).$$

This can be proved by contradiction. Assume that

$$(u^-(t), v^-(t)) \not\equiv (u^+(t), v^+(t)).$$

Then we have

$$u^-(t) < u^+(t), \quad v^-(t) > v^+(t) \quad \forall t \in \mathbb{R}.$$

Observe that

$$\frac{d}{dt} \ln u^\pm(t) = a_1(t) - b_1(t)u^\pm(t) - c_1(t)v^\pm(t) + \frac{d_1(t)}{u^\pm(t)}$$

and

$$\frac{d}{dt} \ln v^\pm(t) = a_2(t) - b_2(t)u^\pm(t) - c_2(t)v^\pm(t) + \frac{d_2(t)}{v^\pm(t)}.$$

Hence

$$\frac{d}{dt} \ln \frac{u^-(t)}{u^+(t)} = b_1(t)[u^+(t) - u^-(t)] + c_1(t)[v^+(t) - v^-(t)] + d_1(t) \left[\frac{1}{u^-(t)} - \frac{1}{u^+(t)} \right]$$

and

$$\frac{d}{dt} \ln \frac{v^-(t)}{v^+(t)} = b_2(t)[u^+(t) - u^-(t)] + c_2(t)[v^+(t) - v^-(t)] + d_2(t) \left[\frac{1}{v^-(t)} - \frac{1}{v^+(t)} \right].$$

It then follows that

$$0 = \int_0^T \frac{d}{dt} \ln \frac{u^-(t)}{u^+(t)} dt > \int_0^T \left[b_1(t)[u^+(t) - u^-(t)] + c_1(t)[v^+(t) - v^-(t)] \right] dt$$

and

$$0 = \int_0^T \frac{d}{dt} \ln \frac{v^-(t)}{v^+(t)} dt < \int_0^T \left[b_2(t)[u^+(t) - u^-(t)] + c_2(t)[v^+(t) - v^-(t)] \right] dt.$$

This implies that

$$b_{1L} \int_0^T [u^+(t) - u^-(t)] dt < c_{1M} \int_0^T [v^-(t) - v^+(t)] dt$$

and

$$b_{2M} \int_0^T [u^+(t) - u^-(t)] dt > c_{1L} \int_0^T [v^-(t) - v^+(t)] dt.$$

Hence

$$\frac{b_{1L}}{c_{1M}} < \frac{\int_0^T [v^-(t) - v^+(t)] dt}{\int_0^T [u^+(t) - u^-(t)] dt} < \frac{b_{2M}}{c_{2L}}.$$

This is a contradiction. The theorem is thus proved. \square

Proof of Theorem 3.1. We prove the theorem for (1.3). It can be proved similarly for (1.4) and (1.5).

For any given $x \in \bar{D}$, let $d_1(t, x) = \int_D k(y-x)u^{**}(t, y)dy$ and $d_2(t, x) = \int_D k(y-x)v^{**}(t, y)dy$. Then $d_1(t, x)$ and $d_2(t, x)$ are positive, periodic in t with period T , and smooth in x . For given $x \in \bar{D}$, $(u(t; x), v(t; x)) = (u^{**}(t, x), v^{**}(t, x))$ satisfies the following competitive systems of ODEs,

$$\begin{cases} u_t(t) = u(t) \left(-\nu_1 + a_1(t, x) - b_1(t, x)u(t) - c_1(t, x)v(t) \right) + d_1(t, x) \\ v_t(t) = v(t) \left(-\nu_2 + a_2(t, x) - b_2(t, x)u(t) - c_2(t, x)v(t) \right) + d_2(t, x). \end{cases} \quad (3.2)$$

By Lemma 3.1, (3.2) has a unique stable time periodic coexistence state $(\tilde{u}^{**}(t; x), \tilde{v}^{**}(t; x))$. By the smoothness of $a_i(t, x)$, $b_i(t, x)$, $c_i(t, x)$, and $d_i(t, x)$ in x for $i = 1, 2$, we have that $(\tilde{u}^{**}(t; x), \tilde{v}^{**}(t; x))$ is continuous in x . Therefore, $(u^{**}(t, x), v^{**}(t, x)) = (\tilde{u}^{**}(t; x), \tilde{v}^{**}(t; x))$ is continuous in x and the theorem then follows. \square

We now prove Theorem A.

Proof of Theorem A. We prove the theorem for (1.3) by applying Theorem 3.1 and modifying the arguments in [17, Theorem A]. It can be proved similarly for (1.4) and (1.5).

(1) Let $(u^*(\cdot, \cdot), 0)$ and $(0, v^*(\cdot, \cdot))$ be the semitrivial time periodic solutions of (1.3). Let $K, I : X_1 \rightarrow X_1$ be given by

$$(Ku)(x) = \int_D \kappa(y-x)u(y)dy, \quad (Iu)(x) = u(x) \quad \forall u \in X_1.$$

First, note that

$$u_t^*(t, x) = \nu_1[K - I]u^*(t, x) + (a_1(t, x) - b_1(t, x)u^*(t, x))u^*(t, x)$$

and

$$u^*(t, x) \leq \frac{a_{1M}}{b_{1L}}.$$

We then have

$$\begin{aligned} a_2(t, x) - b_2(t, x)u^*(t, x) &\geq a_2(t, x) - b_2(t, x)\frac{a_{1M}}{b_{1L}} \\ &\geq a_{2L} - \frac{b_{2M}a_{1M}}{b_{1L}} \\ &> -\nu_2\lambda_0^D. \end{aligned}$$

Note that $\lambda_1(\nu_2, a_{2L} - \frac{b_{2M}a_{1M}}{b_{1L}}) > 0$. By Proposition 2.2, $\lambda := \lambda_1(\nu_2, a_{2L} - \frac{b_{2M}a_{1M}}{b_{1L}})$ is the principal eigenvalue of $\nu_2[K - I]u + [a_{2L} - \frac{b_{2M}a_{1M}}{b_{1L}}]I$. Let $\phi^*(x)$ be a positive principal eigenfunction of

$$\nu_2[K - I]u + [a_{2L} - \frac{b_{2M}a_{1M}}{b_{1L}}]u = \lambda u.$$

Let $v_\epsilon^+(t, x) = \epsilon\phi^*$ and $u_\epsilon^+(t, x) \equiv u^*(t, x)$. We have

$$\begin{cases} (u_\epsilon^+)_t \geq \nu_1[K - I]u_\epsilon^+ + u_\epsilon^+(a_1(t, x) - b_1(t, x)u_\epsilon^+ - c_1(t, x)v_\epsilon^+) \\ (v_\epsilon^+)_t \leq \nu_2[K - I]v_\epsilon^+ + v_\epsilon^+(a_2(t, x) - b_2(t, x)u_\epsilon^+ - c_2(t, x)v_\epsilon^+) \end{cases}$$

for $0 < \epsilon \ll 1$. Hence $(u_\epsilon^+(t, x), v_\epsilon^+(t, x))$ is a super-solution of (1.3). This implies that

$$\begin{aligned} (0, v^*(t, \cdot)) &\leq_2 (u(t + n_2T, \cdot; u^*(0, \cdot), \epsilon\phi^*), v(t + n_2T, \cdot; u^*(0, \cdot), \epsilon\phi^*)) \\ &\leq_2 (u(t + n_1T, \cdot; u^*(0, \cdot), \epsilon\phi^*), v(t + n_1T, \cdot; u^*(0, \cdot), \epsilon\phi^*)) \\ &\leq_2 (u^*(t, \cdot), \epsilon\phi^*) \end{aligned}$$

for any $t \geq 0$ and positive integers $n_2 > n_1$. Hence there are Lebesgue measurable functions $u_{+, \epsilon}^{**}, v_{+, \epsilon}^{**} : \bar{\mathbb{R}}^+ \times D \rightarrow [0, \infty)$ such that

$$(u(t + nT, x; u^*, \epsilon\phi^*), v(t + nT, x; u^*, \epsilon\phi^*)) \rightarrow (u_{+, \epsilon}^{**}(t, x), v_{+, \epsilon}^{**}(t, x)) \quad \forall t \geq 0, x \in \bar{D}$$

as $n \rightarrow \infty$. Moreover,

$$u_{+, \epsilon}^{**}(t+T, x) = u_{+, \epsilon}^{**}(t, x), \quad v_{+, \epsilon}^{**}(t+T, x) = v_{+, \epsilon}^{**}(t, x),$$

and

$$0 \leq u_{+, \epsilon}^{**}(t, x) \leq u^*(t, x), \quad \epsilon \phi^*(x) \leq v_{+, \epsilon}^{**}(t, x) \quad \forall t \geq 0, x \in \bar{D}.$$

Note that

$$\begin{aligned} & u(t+nT, x; u^*(0, \cdot), \epsilon \phi^*) \\ &= u(nT, x; u^*(0, \cdot), \epsilon \phi^*) \\ &+ \nu_1 \int_0^t \left[\int_D \kappa(y-x) u(nT+\tau, y; u^*(0, \cdot), \epsilon \phi^*) dy - u(nT+\tau, x; u^*(0, \cdot), \epsilon \phi^*) \right] d\tau \\ &+ \int_0^t \left[u(nT+\tau, x; u^*(0, \cdot), \epsilon \phi^*) (a_1(\tau, x) - b_1(\tau, x) u(nT+\tau, x; u^*(0, \cdot), \epsilon \phi^*) \right. \\ &\quad \left. - c_1(\tau, x) v(nT+\tau, x; u^*(0, \cdot), \epsilon \phi^*)) \right] d\tau \end{aligned}$$

and

$$\begin{aligned} & v(nT+t, x; u^*(0, \cdot), \epsilon \phi^*) \\ &= v(nT, x; u^*(0, \cdot), \epsilon \phi^*) \\ &+ \nu_2 \int_0^t \left[\int_D \kappa(y-x) v(nT+\tau, y; u^*(0, \cdot), \epsilon \phi^*) dy - v(nT+\tau, x; u^*(0, \cdot), \epsilon \phi^*) \right] d\tau \\ &+ \int_0^t \left[v(nT+\tau, x; u^*(0, \cdot), \epsilon \phi^*) (a_2(\tau, x) - b_2(\tau, x) u(nT+\tau, x; u^*(0, \cdot), \epsilon \phi^*) \right. \\ &\quad \left. - c_2(\tau, x) v(nT+\tau, x; u^*(0, \cdot), \epsilon \phi^*)) \right] d\tau \end{aligned}$$

for any $t > 0$. $n \in \mathbb{N}$, and $x \in \bar{D}$. Letting $n \rightarrow \infty$, by Lebesgue dominating convergent theorem,

$$\begin{aligned} u_{+, \epsilon}^{**}(t, x) &= u_{+, \epsilon}^{**}(0, x) + \nu_1 \int_0^t \left[\int_D \kappa(y-x) u_{+, \epsilon}^{**}(\tau, y) dy - u_{+, \epsilon}^{**}(\tau, x) \right. \\ &\quad \left. + u_{+, \epsilon}^{**}(\tau, x) (a_1(\tau, x) - b_1(\tau, x) u_{+, \epsilon}^{**}(\tau, x) - c_1(\tau, x) v_{+, \epsilon}^{**}(\tau, x)) \right] d\tau \end{aligned}$$

$$\begin{aligned} v_{+, \epsilon}^{**}(t, x) &= v_{+, \epsilon}^{**}(0, x) + \nu_2 \int_0^t \left[\int_D \kappa(y-x) v_{+, \epsilon}^{**}(\tau, y) dy - v_{+, \epsilon}^{**}(\tau, x) \right. \\ &\quad \left. + v_{+, \epsilon}^{**}(\tau, x) (a_2(\tau, x) - b_2(\tau, x) u_{+, \epsilon}^{**}(\tau, x) - c_2(\tau, x) v_{+, \epsilon}^{**}(\tau, x)) \right] d\tau \end{aligned}$$

for all $t > 0$ and $x \in \bar{D}$. It then follows that $(u_{+, \epsilon}^{**}(t, x), v_{+, \epsilon}^{**}(t, x))$ is differentiable in t and satisfies (1.3).

Similarly, let ψ^* be a positive principal eigenfunction of

$$\nu_1[K - I]u + [a_{1L} - \frac{c_{1M}a_{2M}}{c_{2L}}]u = \lambda u,$$

where $\lambda := \lambda_1(\nu_1, a_{1L} - \frac{c_{1M}a_{2M}}{c_{2L}})$. We have that for $0 < \epsilon \ll 1$, there are Lebesgue measurable functions $u_{-, \epsilon}^{**}, v_{-, \epsilon}^{**} : \bar{\mathbb{R}}^+ \times \bar{D} \rightarrow [0, \infty)$ such that

$$(u(nT + t, x; \epsilon \psi^*, v^*(0, \cdot)), v(nT + t, x; \epsilon \psi^*, v^*(0, \cdot))) \rightarrow (u_{-, \epsilon}^{**}(t, x), v_{-, \epsilon}^{**}(t, x)) \quad \forall t \geq 0, x \in \bar{D}$$

as $n \rightarrow \infty$,

$$u_{-, \epsilon}^{**}(t + T, x) = u_{-, \epsilon}^{**}(t, x), \quad v_{-, \epsilon}^{**}(t + T, x) = v_{-, \epsilon}^{**}(t, x),$$

and

$$\epsilon \psi^*(x) \leq u_{-, \epsilon}^{**}(t, x), \quad 0 \leq v_{-, \epsilon}^{**}(t, x) \leq v^*(t, x) \quad \forall x \in \bar{D}.$$

By similar arguments as above, $(u_{-, \epsilon}^{**}(t, x), v_{-, \epsilon}^{**}(t, x))$ is differentiable in t and satisfies (1.3).

Observe that for $0 < \epsilon \ll 1$,

$$\epsilon \psi^*(x) \leq u_{-, \epsilon}^{**}(t, x) \leq u_{+, \epsilon}^{**}(t, x) \leq u^*(t, x), \quad \epsilon \phi^*(x) \leq v_{+, \epsilon}^{**}(t, x) \leq v_{-, \epsilon}^{**}(t, x) \leq v^*(t, x)$$

for all $t \geq 0$ and $x \in \bar{D}$. From $a_{1L} > -\nu_1 \lambda_0^D + \frac{c_{1M}a_{2M}}{c_{2L}}$ and $a_{2L} > -\nu_2 \lambda_0^D + \frac{b_{2M}a_{1M}}{b_{1L}}$ (note that $\lambda_0^D < 0$), we have $\frac{b_{1L}}{c_{1M}} > \frac{b_{2M}}{c_{2L}}$. By Theorem 3.1, both $(u_{-, \epsilon}^{**}(t, x), v_{-, \epsilon}^{**}(t, x))$ and $(u_{+, \epsilon}^{**}(t, x), v_{+, \epsilon}^{**}(t, x))$ are in $\text{Int}X^+ \times \text{Int}X^+$ and hence are coexistence states of (1.3).

(2) Let $(u^*(\cdot, \cdot), 0)$ and $(0, v^*(\cdot, \cdot))$ be the semitrivial time periodic solutions of (1.3). Let $\nu = \nu_1 (= \nu_2)$ and $a(t, x) = a_1(t, x) (= a_2(t, x))$ for $x \in \bar{D}$. Note that

$$u_t^*(t, x) = \nu[K - I]u^*(t, x) + (a(t, x) - b_1(t, x)u^*(t, x))u^*(t, x). \quad (3.3)$$

By $\sup_{x \in \bar{D}} b_2(t, x) < \inf_{x \in \bar{D}} b_1(t, x)$ for any $t \in \mathbb{R}$, we have $b_2(t, x) < b_1(t, x)$ for $t \in \mathbb{R}$ and $x \in \bar{D}$. Then

$$a(t, x) - b_2(t, x)u^*(t, x) > a(t, x) - b_1(t, x)u^*(t, x) \quad \forall t \in \mathbb{R} \quad x \in \bar{D}.$$

Let

$$\epsilon_+^* = \inf_{t \in \mathbb{R}, x \in \bar{D}} (b_1(t, x) - b_2(t, x))u^*(t, x) (> 0).$$

Then

$$a(t, x) - b_2(t, x)u^*(t, x) > a(t, x) - b_1(t, x)u^*(t, x) + \frac{\epsilon_+^*}{2} \quad \forall t \in \mathbb{R}, x \in \bar{D}.$$

Hence $v_\epsilon^+(t, x) = \epsilon u^*(t, x)$ ($0 < \epsilon \ll 1$) is a strictly sub-solution of

$$v_t = \nu[K - I]v + (a(t, x) - b_2(t, x)u^*(t, x))v.$$

By the similar arguments as in (1), for $0 < \epsilon \ll 1$, there are Lebesgue measurable functions $u_{+, \epsilon}^{**}, v_{+, \epsilon}^{**} : \bar{\mathbb{R}}^+ \times \bar{D} \rightarrow [0, \infty)$ such that

$$(u(nT + t, x; u^*(0, \cdot), \epsilon u^*(0, \cdot)), v(nT + t, x; u^*(0, \cdot), \epsilon u^*(0, \cdot))) \rightarrow (u_{+, \epsilon}^{**}(t, x), v_{+, \epsilon}^{**}(t, x)) \quad \forall t \geq 0, x \in \bar{D}$$

as $n \rightarrow \infty$,

$$u_{+, \epsilon}^{**}(t + T, x) = u_{+, \epsilon}^{**}(t, x), \quad v_{+, \epsilon}^{**}(t + T, x) = v_{+, \epsilon}^{**}(t, x),$$

and $(u_{+, \epsilon}^{**}(t, x), v_{+, \epsilon}^{**}(t, x))$ satisfies (1.3).

Similarly, by $\inf_{x \in \bar{D}} c_2(t, x) > \sup_{x \in \bar{D}} c_1(t, x)$ for all $t \in \mathbb{R}$, we have

$$a(t, x) - c_1(t, x)v^*(t, x) > a(t, x) - c_2(t, x)v^*(t, x).$$

Set

$$\epsilon_-^* = \inf_{t \in \mathbb{R}, x \in \bar{D}} (c_2(t, x) - c_1(t, x))v^*(t, x) (> 0),$$

then

$$a(t, x) - c_1(t, x)v^*(t, x) > a(t, x) - c_2(t, x)c^*(t, x) + \frac{\epsilon_-^*}{2}.$$

Thus, given $0 < \epsilon \ll 1$, there are Lebesgue measurable functions $u_{-, \epsilon}^{**}, v_{-, \epsilon}^{**} : \mathbb{R}^+ \times \bar{D} \rightarrow [0, \infty)$ such that

$$(u(t+nT, x; \epsilon v^*(0, \cdot), v^*(0, \cdot)), v(t+nT, x; \epsilon v^*(0, \cdot), v^*(0, \cdot))) \rightarrow (u_{-, \epsilon}^{**}(t, x), v_{-, \epsilon}^{**}(t, x)) \quad \forall t \geq 0, x \in \bar{D}$$

as $n \rightarrow \infty$,

$$u_{-, \epsilon}^{**}(t+T, x) = u_{-, \epsilon}^{**}(t, x), \quad v_{-, \epsilon}^{**}(t+T, x) = v_{-, \epsilon}^{**}(t, x),$$

and $(u_{-, \epsilon}^{**}(t, x), v_{-, \epsilon}^{**}(t, x))$ satisfies (1.3).

Then by the similar arguments as in (1), $(u_{\pm, \epsilon}^{**}(t, x), v_{\pm, \epsilon}^{**}(t, x))$ belongs to $\text{Int}X^+ \times \text{Int}X^+$ and hence are coexistence states of (1.3).

(3) It is a special case of (2). By (2), (1.3) has coexistence states. We first prove that the coexistence state of (1.3) is unique.

Let $(u^{**}(t, x), v^{**}(t, x))$ be any given coexistence state of (1.3). Put $\nu = \nu_1 (= \nu_2)$ and $a(\cdot, \cdot) = a_1(\cdot, \cdot) (= a_2(\cdot, \cdot))$. Then

$$\begin{cases} u_t^{**}(t, x) = \nu[K - I]u^{**} + u^{**}(a(t, x) - b_1u^{**} - c_2v^{**}) + (c_2 - c_1)u^{**}v^{**}, & x \in \bar{D} \\ v_t^{**}(t, x) = \nu[K - I]v^{**} + v^{**}(a(t, x) - b_1u^{**} - c_2v^{**}) + (b_1 - b_2)u^{**}v^{**}, & x \in \bar{D}. \end{cases}$$

Multiplying the first equation by $(b_1 - b_2)$ and the second one by $(c_2 - c_1)$, we obtain

$$\begin{aligned} & -(b_1 - b_2)u_t^{**}(t, x) + (b_1 - b_2)\nu[K - I]u^{**} + (b_1 - b_2)u^{**}(a(t, x) - b_1u^{**} - c_2v^{**}) \\ & = -(c_2 - c_1)v_t^{**}(t, x) + (c_2 - c_1)\nu[K - I]v^{**} + (c_2 - c_1)v^{**}(a(t, x) - b_1u^{**} - c_2v^{**}). \end{aligned}$$

This implies that

$$\phi_t^{**}(t, x) = \nu[K - I]\phi^{**} + (a(t, x) - b_1u^{**} - c_2v^{**})\phi^{**}, \quad (3.4)$$

where $\phi^{**}(t, x) = (b_1 - b_2)u^{**}(t, x) - (c_2 - c_1)v^{**}(t, x)$. Observe that

$$a(t, x) - b_1u^{**}(t, x) - c_2v^{**}(t, x) < a(t, x) - b_1u^{**}(t, x) - c_1v^{**}(t, x), \quad (3.5)$$

$$u_t^{**}(t, x) = \nu[K - I]u^{**} + (a(t, x) - b_1u^{**} - c_1v^{**})u^{**}. \quad (3.6)$$

By (3.6) and Proposition 2.4, $\lambda(\nu, a(\cdot, \cdot) - b_1 u^{**} - c_1 v^{**})$ exists and $\lambda(\nu, a(\cdot, \cdot) - b_1 u^{**} - c_1 v^{**}) = 0$. This together with (3.5) implies that (3.4) has only the trivial solution. Therefore $\phi^{**} \equiv 0$, that is,

$$v^{**} = \frac{b_1 - b_2}{c_2 - c_1} u^{**}. \quad (3.7)$$

By (3.7), u^{**} is the unique positive solution of

$$u_t^{**} = \nu[K - I]u^{**} + \left[a(t, x) - (b + c_1 \cdot \frac{b_1 - b_2}{c_2 - c_1})u^{**} \right] u^{**}. \quad (3.8)$$

By (3.7) and (3.8), the coexistence state of (1.3) is unique.

Next we prove the global stability of the unique coexistence state (u^{**}, v^{**}) . Let θ^* be the unique time periodic positive solution of

$$u_t = \nu[K - I]u + u(a(t, x) - u) \quad (3.9)$$

(see [30, Theorem E] for the existence of θ^*). Then $u^* = \frac{\theta^*}{b_1}$ and $v^* = \frac{\theta^*}{c_2}$.

For $\alpha_+, \beta_+ > 0$ with $\frac{1}{b_1} < \alpha_+ < \frac{1}{b_2}$ and $0 < \beta_+ \ll 1$, let $u_+ = \alpha_+ \theta^*$ and $v_+ = \beta_+ \theta^*$. We then have

$$\begin{cases} (u_+)_t \geq \nu[K - I]u_+ + u_+(a(t, x) - b_1 u_+ - c_1 v_+), & x \in \bar{D} \\ (v_+)_t \leq \nu[K - I]v_+ + v_+(a(t, x) - b_2 u_+ - c_2 v_+), & x \in \bar{D}. \end{cases}$$

Therefore,

$$\begin{aligned} u(t + n_2 T, \cdot; u_+, v_+), v(t + n_2 T, \cdot; u_+, v_+) &\leq_2 (u(t + n_1 T, \cdot; u_+, v_+), v(t + n_1 T, \cdot; u_+, v_+)) \\ &\leq_2 (u_+(t, \cdot), v_+(t, \cdot)) \end{aligned}$$

for every $t \geq 0$ and any positive integers $n_2 > n_1$. This implies that

$$u(t, \cdot; u_+, v_+), v(t, \cdot; u_+, v_+) - (u^{**}(t, x), v^{**}(t, x)) \rightarrow (0, 0) \text{ as } t \rightarrow \infty.$$

Similarly, for $\alpha_-, \beta_- > 0$ with $\frac{1}{c_2} < \beta_- < \frac{1}{c_1}$ and $0 < \alpha_- \ll 1$, let $u_- = \alpha_- \theta^*$ and $v_- = \beta_- \theta^*$. Then

$$\begin{aligned} u(t + n_2 T, \cdot; u_-, v_-), v(t + n_2 T, \cdot; u_-, v_-) &\geq_2 (u(t + n_1 T, \cdot; u_-, v_-), v(t + n_1 T, \cdot; u_-, v_-)) \\ &\geq_2 (u_-(t, \cdot), v_-(t, \cdot)) \end{aligned}$$

for every $t \geq 0$ and any positive integers $n_2 > n_1$, thus

$$u(t, \cdot; u_-, v_-), v(t, \cdot; u_-, v_-) - (u^{**}(t, \cdot), v^{**}(t, \cdot)) \rightarrow (0, 0) \text{ as } t \rightarrow \infty.$$

For any given $(u_0, v_0) \in (X^+ \setminus \{0\}) \times (X^+ \setminus \{0\})$ and any $\epsilon > 0$, by Proposition 2.5, there is $n \in \mathbb{N}$ such that

$$(0, v^* + \epsilon) \ll_2 (u(t + nT, \cdot; u_0, v_0), v(t + nT, \cdot; u_0, v_0)) \ll_2 (u^* + \epsilon, 0)$$

for $t \geq 0$. Then there are $\alpha_{\pm}, \beta_{\pm} > 0$ with $\frac{1}{b_1} < \alpha_+ < \frac{1}{b_2}$, $0 < \beta_+ \ll 1$, and $\frac{1}{c_2} < \beta_- < \frac{1}{c_1}$, $0 < \alpha_- \ll 1$ such that

$$(\alpha_- \theta^*, \beta_- \theta^*) \leq_2 (u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) \leq_2 (\alpha_+ \theta^*, \beta_+ \theta^*)$$

for $t \gg 1$. It therefore follows that

$$(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) - (u^{**}(t, \cdot), v^{**}(t, \cdot)) \rightarrow (0, 0)$$

as $t \rightarrow \infty$. Theorem A is thus proved. \square

4 Extinction

In this section, we study the extinction dynamics of (1.3), (1.4), and (1.5), and prove Theorem B. Let $(u^*(t, x), 0)$ and $(0, v^*(t, x))$ be the two semitrivial periodic solutions of (1.3) (resp. (1.4), (1.5)). We say that $(u^*, 0)$ (resp. $(0, v^*)$) is *globally stable* if for any $(u_0, v_0) \in (X_1^+ \setminus \{0\}) \times (X_1^+ \setminus \{0\})$ (resp. $(u_0, v_0) \in (X_2^+ \setminus \{0\}) \times (X_2^+ \setminus \{0\})$, $(u_0, v_0) \in (X_3^+ \setminus \{0\}) \times (X_3^+ \setminus \{0\})$),

$$(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) - (u^*(t, \cdot), 0) \rightarrow (0, 0)$$

$$(\text{resp. } (u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) - (0, v^*(t, \cdot)) \rightarrow (0, 0))$$

as $t \rightarrow \infty$, where $(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0))$ is the solution of (1.3) (resp. (1.4), (1.5)) with initial (u_0, v_0) .

Proof of Theorem B. We prove Theorem for (1.3). It can be proved similarly for (1.4), and (1.5).

(1) First consider

$$\begin{cases} u_t = \nu[K - I]u + u(a_{1L} - b_{1M}u - c_{1M}v), & x \in \bar{D} \\ v_t = \nu[K - I]v + v(a_{2M} - b_{2L}u - c_{2L}v), & x \in \bar{D}. \end{cases} \quad (4.1)$$

For any given $(u_0, v_0) \in X_1^+ \times X_1^+$, let $(u^-(t, x; u_0, v_0), v^-(t, x; u_0, v_0))$ be the solution of (4.1) with $(u^-(0, x; u_0, v_0), v^-(0, x; u_0, v_0)) = (u_0(x), v_0(x))$. Let $(u_-^*, 0)$ and $(0, v_-^*)$ be the semitrivial equilibria of (4.1).

By the arguments of [17, Theorem B], for any $(u_0, v_0) \in (X_1^+ \setminus \{0\}) \times (X_1^+ \setminus \{0\})$,

$$\lim_{t \rightarrow \infty} (u^-(t, \cdot; u_0, v_0), v^-(t, \cdot; u_0, v_0)) = (u_-^*, 0).$$

For any given $(u_0, v_0) \in (X_1^+ \setminus \{0\}) \times (X_1^+ \setminus \{0\})$, for any $\epsilon_2 > 0$, by Proposition 2.5, there is $\tilde{T} > 0$ such that

$$(0, (1 + \epsilon_2)v_-^*) \ll_2 (u^-(t, \cdot; u_0, v_0), v^-(t, \cdot; u_0, v_0))$$

for $t \geq \tilde{T}$. Then there is $\epsilon_1 > 0$ such that

$$(\epsilon_1 v_-^*, (1 + \epsilon_2)v_-^*) \leq_2 (u^-(t, \cdot; u_0, v_0), v^-(t, \cdot; u_0, v_0))$$

for $t \gg 1$. Thus we have

$$(u^-(t, \cdot; u_0, v_0), v^-(t, \cdot; u_0, v_0)) \rightarrow (u_-^*, 0)$$

as $t \rightarrow \infty$, and the claim is proved.

For any given $(u_0, v_0) \in X_1^+ \times X_1^+$, by Proposition 2.6,

$$(u^-(t, \cdot; u_0, v_0), v^-(t, \cdot; u_0, v_0)) \leq_2 (u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) \quad \forall t > 0.$$

By the above claim, for any $(u_0, v_0) \in (X_1^+ \setminus \{0\}) \times (X_1^+ \setminus \{0\})$,

$$\lim_{t \rightarrow \infty} (u^-(t, \cdot; u_0, v_0), v^-(t, \cdot; u_0, v_0)) = (u_-^*, 0).$$

This together with Proposition 2.5 implies that

$$\lim_{t \rightarrow \infty} [(u(t, \cdot; u_0, v_0), v(t, \cdot; u_0, v_0)) - (u^*(t, \cdot), 0)] = (0, 0).$$

(2) can be proved by the similar arguments as in (1). □

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